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Optimal Spherical Designs and Numerical Integration on the Sphere

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So-called spherical t-designs in three-dimensional Euclidean space $\mathbb{R}^3$ are considered. It is shown that the maximal possible value of t for designs which are permuted transitively under a finite group of Euclidean transformations is 9. This means that we can find a set of (60) points on the surface of the unit sphere in $\mathbb{R}^3$ which form an orbit under the icosahedral group such that the average of the function values at the 60 points and the average over the spherical surface are identical for all spherical harmonics of degree less than or equal to $t = 9$, while there is no such orbit for $t = 10$. Among all 9-designs which are orbits of the icosahedral group exactly one is optimal with respect to approximation of integrals of harmonics of tenth and eleventh order. This set is given in Section 2 of this paper. It is very well suited for numerically evaluating integrals of continuous functions over the sphere, a problem occurring very often in mathematics and science, as, e.g., physics, astronomy, etc.

INTRODUCTION

In mathematics and science one quite often has to integrate a function which is known numerically over a closed surface. For instance, the author was led to this problem in the context of an astrophysical investigation. To determine the structure of the heliopause, a contact discontinuity between the solar wind and interstellar matter, it is necessary to evaluate certain integrals of the heliopause pressure over the surface [1].

If the surface is homeomorphic to the sphere, a reduction to the special case of integration over the unit sphere suggests itself. This may be effected by defining a number of selected points on the sphere and approximating the integral by a (weighted) sum over the function values at these points. It is useful to distribute the points as symmetrically as possible, i.e., to choose them in such a way that all of them are permuted transitively by a finite group of isometries of the sphere (and to give equal weights to them).

In this paper it will be shown that there is—up to isomorphism— just one of these sets which is optimal in the following sense: It allows one to integrate exactly all spherical harmonics of degree less than 10 while any other set with the above-mentioned symmetry property gives a worse approximation for integrals of some 10-order harmonics.

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In particular, it is found that no finite set of points on the sphere satisfies the requirement of symmetry and integrates all 10-order harmonics exactly.

An analogous investigation of the four-dimensional case is in preparation.

1. Definitions and Notation

First we have to introduce some notation. For any given positive integer n we denote by $\mathbb{R}^n$ the Euclidean space of dimension n with the usual scalar product $x \cdot y$ and the induced metric. Let Ω_n be the set of unit vectors (the "sphere")

$$\Omega_n = \{x = (x_1, \dots, x_n) \mid |x|^2 = x_1^2 + \cdots + x_n^2 = 1\}. \tag{1}$$

The set of all Euclidean transformations leaving the zero vector o invariant forms the orthogonal group O_n (including reflections). Of course, Ω_n is invariant under O_n.

For any continuous function $f\colon \Omega_n \to \mathbb{R}$ we define the (integral) average $\langle f \rangle_{\Omega_n}$ by

$$\langle f \rangle_{\Omega_n} = (\operatorname{vol} \Omega_n)^{-1} \underset{x_1^2 + \cdots + x_n^2 = 1}{\int_{x_1} \cdots \int_{x_n}} f(x_1, \dots, x_n)\, dO_{(x_1, \dots, x_n)} = (\operatorname{vol} \Omega_n)^{-1} \int_{x \in \Omega_n} f(x)\, dO_x, \tag{2}$$

where $f(x)$ means $f(x_1, \dots, x_n)$, dO_x is the (Euclidean) surface element at x and

$$\operatorname{vol} \Omega_n = \int_{x \in \Omega_n} dO_x = \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{3}$$

is the area of the sphere Ω_n.

If $M \neq \varnothing$ is a *finite* subset of Ω_n, we may also define the M-average $\langle f \rangle_M$ of f:

$$\langle f \rangle_M = (\# M)^{-1} \sum_{x \in M} f(x). \tag{4}$$

M is said to integrate f exactly if the M-average and the integral average of f coincide:

$$\langle f \rangle_M = \langle f \rangle_{\Omega_n}. \tag{5}$$

Let the space of n-variable-polynomials of degree $\leqslant k$ be denoted by $\operatorname{Pol}(n, k)$.

DEFINITION 1 (Delsarte *et al.* [2]). *A finite subset M of Ω_n is called an n-dimensional (spherical) t-design for $t \in \mathbb{N}$ if every polynomial P of degree $\leqslant t$ is integrated exactly by M*:

$$\bigwedge_{P \in \operatorname{Pol}(n,k)} \langle P \rangle_M = \langle P \rangle_{\Omega_n}. \tag{6}$$

(It should be remembered that the spherical harmonics of degree $\leqslant k$ are just the restrictions of the $\mathrm{Pol}(n, k)$-polynomials to Ω_n.)

In the following we shall need a subspace of $\mathrm{Pol}(n, k)$, namely, the space of all homogeneous polynomials of degree k:

$$\mathrm{Hom}(n, k) = \{P \in \mathrm{Pol}(n, k) \mid x \cdot \nabla_x P = k \cdot P\}. \tag{7}$$

Here ∇ is the gradient operator

$$\nabla_x P = (\partial_{x_1} P, \ldots, \partial_{x_n} P). \tag{8}$$

It is well-known that for any given polynomial $P \in \mathrm{Pol}(n, k)$ there is exactly one harmonic polynomial H such that the restrictions of P and H are identical on Ω_n:

$$\bigwedge_{x \in \Omega_n} P(x) = H(x) \tag{9}$$

and the degree of H is not larger than the degree of P. A direct consequence of this fact is

LEMMA 1 (cf. Delsarte *et al.*. [2]; Bannai [3]. *The number of points of a spherical n-dimensional t-design M is bounded below by*

$$\#M \geqslant \binom{n+s-1}{n-1} + \binom{n+s-2}{n-1} \qquad \text{if } t = 2s \text{ is even}, \tag{10}$$

$$\#M \geqslant 2\binom{n+s-1}{n-1} \qquad \text{if } t = 2s+1 \text{ is odd}. \tag{11}$$

The theory of the so-called tight t-designs, for which this bound is attained, is now developed quite well. Bannai and Damerell [4] have shown that tight designs in $\mathbb{R}^n$ with $n \geqslant 3$ can (and do) exist only for $t = 1, 2, 3, 4, 5, 7$ and 11. Two spectacular examples are the minimal vectors of the E_8-lattice ($n = 8$, $t = 7$) and the Leech lattice ($n = 24$, $t = 11$).

Tight designs are most "economical" in that they merely contain the minimum necessary number of points. But for practical applications (integration) the dimension n is prescribed in advance and we have to search for t-designs with t as large as possible. As the referees informed me, Seymour and Zaslavsky [10] announced a proof that spherical t-designs of dimension n exist for all n and t while, on the other hand, Bannai [5] recently showed that t is bounded above (by about 19 to 23) if one demands that the points of the design are permuted transitively by a finite orthogonal group. Bannai's proof is said to depend on the recently finished classification of the finite simple groups.

Here we want to investigate a more modest question: Which t-designs exist for a certain value of n with the additional property of being permuted transitively under the action of a finite O_n-subgroup? This leads to

DEFINITION 2. Let G be a finite subgroup of O_n, $t \in \mathbb{N}$. A subset M of Ω_n is called a *t-orbit* of G, if M is an orbit of G:

$$M = x^G = \{x^g \mid g \in G\}; \qquad x \in \Omega_n \tag{12}$$

and M is a t-design.

DEFINITION 3. A t-orbit M of the finite group $G < O_n$ is called *optimal* relative to the polynomial P, if

(a) there is no $(t+1)$-orbit of G and

(b) for any t-orbit M' of G

$$|\langle P\rangle_M - \langle P\rangle_{\Omega_n}| \leqslant |\langle P\rangle_{M'} - \langle P\rangle_{\Omega_n}|. \tag{13}$$

Finally we come to

DEFINITION 4. Let G be a finite subgroup of O_n. For any function $f\colon \Omega_n \to \mathbb{R}$ we introduce a new function f_G by

$$f_G(x) = |G|^{-1} \sum_{g \in G} f(x^g) \qquad \text{for all} \quad x \in \Omega_n. \tag{14}$$

Of course, f_G is invariant under G, and f is invariant if and only if $f = f_G$. Furthermore, if f is a (homogeneous) polynomial, so is f_G. The degree of f_G is less than or equal to the degree of f.

From the definitions we get immediately

PROPOSITION 1. *Let G be a finite subgroup of O_n, M a G-orbit on Ω_n: $M = x^G$, $x \in \Omega_n$. A necessary and sufficient condition for M to be a t-orbit of G is*

$$P(x) = \langle P\rangle_{\Omega_n} \tag{15}$$

for every G-invariant polynomial P of degree $\leqslant t$.

Proof. Obvious because the Euclidean measure is invariant under O_n.

2. THE ICOSAHEDRAL GROUP AND ITS ORBITS

From now on we shall concentrate on the case $n = 3$, which is the most important one for applications.

The finite subgroups of O_3 have been known for some time (cf., e.g., Coxeter [6]). They are

(a) the cyclic groups,

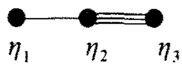
(b) the dihedral groups,

(c) the (extended) octahedral and icosahedral groups, H and I, and their subgroups.

The last two groups may also be described in terms of symmetric and alternating groups:

$$H \simeq 2 \times S_4; \qquad I \simeq 2 \times A_5. \tag{1}$$

The most interesting of all these groups is I. It is defined as the Coxeter group generated by the reflections in the roots of the root system of type H_3. The Coxeter diagram is (cf. Coxeter and Moser [7])

H_3: 

(2)

A generating set of roots is, e.g.,

$$\eta_1 = (1, 0, 0); \qquad \eta_2 = \tfrac{1}{2}(-1, \lambda, \lambda'); \qquad \eta_3 = (0, 0, 1), \tag{3}$$

where

$$\lambda = \tfrac{1}{2}(-1 + \sqrt{5}); \qquad \lambda' = \tfrac{1}{2}(-1 - \sqrt{5}) \tag{4}$$

are the two solutions of the equation

$$s^2 + s = 1. \tag{5}$$

Applying group I to η_1, η_2, η_3 we find exactly 30 roots, namely,

$$(1, 0, 0); \qquad \tfrac{1}{2}(1, \lambda, \lambda') \tag{6}$$

and the vectors obtained from them by cyclically permuting the coordinates and/or changing signs of one or more coordinates; the operations

$$(x_1, x_2, x_3) \mapsto (x_2, x_3, x_1) \tag{7}$$

and

$$(x_1, x_2, x_3) \mapsto (\pm x_1, \pm x_2, \pm x_3) \tag{8}$$

are elements of I.

The (unit) fix vectors of the 6 Sylow-5-subgroups of I,

$$\begin{gathered} \frac{2}{\sqrt{5}}\left(0, \pm\sin\frac{4\pi}{5}, \pm\sin\frac{2\pi}{5}\right); \qquad \frac{2}{\sqrt{5}}\left(\pm\sin\frac{2\pi}{5}, 0, \pm\sin\frac{4\pi}{5}\right); \\ \frac{2}{\sqrt{5}}\left(\pm\sin\frac{4\pi}{5}, \pm\sin\frac{2\pi}{5}, 0\right) \end{gathered} \tag{9}$$

are the 12 corners of a regular icosahedron. Its faces have centres (projected on Ω_3) at the 20 points

$$\begin{aligned} &\frac{1}{\sqrt{3}}(\pm 1, \pm 1, \pm 1); \qquad \frac{1}{\sqrt{3}}(0, \pm\lambda', \pm\lambda); \\ &\frac{1}{\sqrt{3}}(\pm\lambda, 0, \pm\lambda'); \qquad \frac{1}{\sqrt{3}}(\pm\lambda', \pm\lambda, 0) \end{aligned} \tag{10}$$

fixed under one of the 10 Sylow-3-groups in I.

The centres of the 30 edges of the icosahedron are the fixed points of the 15 involutions with determinant 1 in I:

$$\begin{aligned} &(\pm 1, 0, 0); \qquad (0, \pm 1, 0); \qquad (0, 0, \pm 1); \\ &\tfrac{1}{2}(\pm 1, \pm\lambda, \pm\lambda'); \qquad \tfrac{1}{2}(\pm\lambda', \pm 1, \pm\lambda); \qquad \tfrac{1}{2}(\pm\lambda, \lambda', \pm 1). \end{aligned} \tag{11}$$

These are the roots given in (6).

Each of the sets (9), (10) and (11) are permuted transitively under I. They are the only I-orbits of lengths unequal to 60 or 120.

There are three I-invariant homogeneous polynomials of degree 2, 6 and 10, namely,

$$P_2 = P_2(x) = x_1^2 + x_2^2 + x_3^2 \tag{12}$$

(which is even O_3-invariant), the product P_6 (resp. P_{10}) of the planes invariant as a whole under each of the 6 Sylow-5-groups (the 10 Sylow-3-groups) of I. They are—up to a constant factor—uniquely determined,

$$P_6 = P_6(x) = 4x_1^2x_2^2x_3^2 + \lambda(x_1^4x_2^2 + x_2^4x_3^2 + x_3^4x_1^2) + \lambda'(x_1^2x_2^4 + x_2^2x_3^4 + x_3^2x_1^4), \tag{13}$$

$$\begin{aligned} P_{10} = P_{10}(x) = \sqrt{5}\,&[x_1^4 + x_2^4 + x_3^4 - 2x_1^2x_2^2 - 2x_2^2x_3^2 - 2x_3^2x_1^2] \\ &\times [(\lambda'^6 - \lambda^6)\,x_1^2x_2^2x_3^2 + \lambda^2(x_1^2x_2^4 + x_2^2x_3^4 + x_3^2x_1^4) - \lambda'^2(x_1^4x_2^2 + x_2^4x_3^2 + x_3^4x_1^2)]. \end{aligned} \tag{14}$$

The Ω_3-averages are

$$\langle P_6\rangle_{\Omega_3} = -1/21; \qquad \langle P_{10}\rangle_{\Omega_3} = -5/77. \tag{15}$$

I-invariance of the polynomials is obvious and it is also easy to see that there is no nontrivial polynomial in P_2, P_6 and P_{10} vanishing identically, i.e., they are algebraically independent.

Next we evaluate the Molien series of I:

$$M_I(\varepsilon) = |I|^{-1} \sum_{\mu \in I} [\det(1 - \varepsilon\mu)]^{-1} = [(1 - \varepsilon^2)(1 - \varepsilon^6)(1 - \varepsilon^{10})]^{-1}. \tag{16}$$

This shows that I has three basic invariants with degrees 2, 6 and 10. We choose them as P_2, P_6 and P_{10}.

Hence a polynomial is I-invariant if and only if it can be written as a polynomial in P_2, P_6 and P_{10}.

As $P_2(x) = 1$ for all $x \in \Omega_3$, we have by proposition 1:

LEMMA 2. *Let $x \in \Omega_3$. x^I is a t-orbit of I if and only if for every $a, b \geqslant 0$, $(a, b) \neq (0, 0)$ with $10a + 6b \leqslant t$*

$$[P_{10}(x)]^a [P_6(x)]^b = \langle P_{10}^a P_6^b \rangle_{\Omega_3} \tag{17}$$

holds.

Now we are able to formulate the central result of our investigations.

MAIN THEOREM. *For all $x \in \Omega_3$ is x^I a 5-design. If furthermore $P_6(x) = -1/21$, x^I is even a 9-design (there is an infinity of such x). I does not possess a 10-orbit. The 60 vectors given in Table I form a 9-orbit M of I which is optimal relative to every polynomial of degree $\leqslant 11$.*

Proof. Ω_3 is a compact set and P_6 a continuous function, so P_6 attains its mean value which is $\langle P_6 \rangle_{\Omega_3} = -1/21$. Hence by the preceding lemma the first two assertions are proven. All $x \in \Omega_3$ with

$$P_6(x) = \langle P_6 \rangle_{\Omega_3} = -1/21 \tag{18}$$

therefore lie on a curve K which is the intersection of Ω_3 and the surface F with defining equation (18).

Ω_3 is compact, F topologically closed, so $K = \Omega_3 \cap F$ is compact, too. Hence there

TABLE I

The Optimal 9-Orbit of I

$x_1 = 0.7950079322147038205344672$
$x_2 = 0.6065990337246679914486006$
$x_3 = 0.0000000000000000000000000$
$y_1 = 0.8882528931620301775150026$
$y_2 = 0.4557255176322373522904353$
$y_3 = 0.0576285551451042821776299$
$z_1 = 0.8306243380169258953373727$
$z_2 = 0.5489704785795637092709707$
$z_3 = 0.0932449609473263569805355$

Note. I contains the $3 \cdot 4 + 3 \cdot 8 + 3 \cdot 8 = 60$ images of (x_1, x_2, x_3), (y_1, y_2, y_3) and (z_1, z_2, z_3) under cyclic permutations and/or sign changes of the coordinates.

must be points where the restriction $P_{10|K}$ of P_{10} to K is maximal (or minimal). Let $x \in K$ be a regular point with $P_{10|K}$ extremal in x. The condition for this is

$$t_x \cdot \nabla_x P_{10} = 0 \tag{19}$$

with t_x = tangent vector at K in x. t_x is tangent to both surfaces, Ω_3 and F, so it must be orthogonal to $\nabla_x P_2$ and $\nabla_x P_6$ and (19) implies

$$S(x) = (\nabla_x P_2 \wedge \nabla_x P_6) \cdot \nabla_x P_{10} = 0. \tag{20}$$

If, however, x is a singular point of K, either F is singular at x:

$$\nabla_x P_6 = o \tag{21}$$

or the tangent planes of Ω_3 and F in x coincide:

$$\nabla_x P_2 \wedge \nabla_x P_6 = o. \tag{22}$$

Both (21) and (22) lead us back to (20), so we only have to discuss the latter equation for $x \in K$.

There are exactly 15 planes, each containing two opposite edges of the icosahedron. They all are permuted transitively by I. Their product is a homogeneous polynomial $P_{15}(x)$ of degree 15 in x. By definition, the set of all x with

$$P_{15}(x) = 0 \tag{23}$$

is the union of the 15 planes, and so is I-invariant, while P_{15} itself is not (its sign changes if we apply a reflection in I).

The function S on the left-hand side of (20) also is homogeneous of degree $(2-1)+(6-1)+(10-1)=15$ and I-invariant up to sign, because P_2, P_6 and P_{10} are. The latter three polynomials are even functions of x_3; hence the third components of their gradients vanish for $x_3 = 0$, i.e., (20) holds true for all x with $x_3 = 0$.

But $x_3 = 0$ is one of the 15 planes just mentioned, so all of them must be solutions of (20). Thus S and P_{15} are identical up to a nonzero factor, and (20) is equivalent to (23).

For reasons of transitivity on the 15 planes it is sufficient to solve

$$P_2(x) = 1; \qquad P_6(x) = -\frac{1}{21} \tag{24}$$

for x in one of the planes, e.g., for $x_3 = 0$. We introduce the abbreviations

$$u = x_1^2; \qquad v = x_2^2 \qquad (x_3 = 0) \tag{25}$$

and get from (24) the conditions

$$P_2 = u + v = 1 \tag{26}$$

and

$$P_6 = uv(\lambda u + \lambda' v) = -\frac{1}{21}. \tag{27}$$

Elimination of one of the unknowns in (27) with the help of (26) provides us with

$$u^3 - \frac{8+\lambda}{5} u^2 + \frac{3+\lambda}{5} u - \frac{1}{21\sqrt{5}} = 0 \tag{28}$$

and

$$v^3 - \frac{8+\lambda'}{5} v^2 + \frac{3+\lambda'}{5} v + \frac{1}{21\sqrt{5}} = 0. \tag{29}$$

It is easily found that there are just three real pairs (u, v), namely,

$$(u_1, v_1) \approx (0.632, 0.368); \qquad (u_2, v_2) \approx (0.032, 0.968); \qquad (u_3, v_3) \approx (1.060, -0.060) \tag{30}$$

the last of which is not admissible, for by construction of u, v

$$0 \leqslant u,\ v \leqslant 1. \tag{31}$$

One of the remaining couples (u_1, v_1) and (u_2, v_2) must lead to the maximum, the other to the minimum of P_{10} along K.

Numerically we find

$$P_{10}(u_1, v_1) \approx -0.055; \qquad P_{10}(u_2, v_2) \approx 0.017 \tag{32}$$

so the first value is the minimum, the second the maximum (in both cases, four combinations of the square roots $x_1 = \pm\sqrt{u}$, $x_2 = \pm\sqrt{v}$ are possible, but they lie in one I-orbit).

Because even the minimum is greater than the Ω_3-average of P_{10},

$$\langle P_{10} \rangle_{\Omega_3} = -\frac{5}{77} \approx -0.065 \tag{33}$$

we have found that the orbit $M = x^I$ belonging to (u_1, v_1) is a 9-orbit of I optimal with respect to P_{10}, and that there cannot be any other with the same property. In particular, I does not possess a 10-orbit.

To finish the proof, we only have to verify that M is optimal with respect to every polynomial Q of degree at most 11. Clearly by an earlier argument we may restrict our attention to the I-invariant Q's of degree $\leqslant 11$, i.e., to P_6 and P_{10}. Optimality relative to P_{10} was shown above and M as a 9-design clearly is optimal relative to P_6, this proving the theorem. It should be remarked that the design which is derived from the second solution (u_2, v_2) in (30)—the "worst-possible" 9-orbit of I—is mentioned by Goethals and Seidel [8] as the "improved football", but the authors do not consider the question of optimality in the sense defined above, rather they give many other related formulas in a more general context (spherical designs which are not orbits of any group, more complicated cubature formulas, etc.). I am grateful to the unknown referees who directed my attention to this interesting paper.

3. The Other Three-Dimensional Finite Groups

To complete the discussion, we note

THEOREM. *There is no* 9*-orbit for any of the groups listed in Section* 2, *except the icosahedral group I and its commutator group I'.*

Proof. This is obvious for the cyclic and dihedral groups, because their orbits are subsets of at most two planes intersecting the sphere Ω_3.

So we are left with the subgroups of H and I.

A 3-dimensional 9-design must consist of at least 30 vectors by Eq. (1.11).

The only subgroups of H and I which are large enough are H, I and I'. The verification that H does not possess 9-orbits is carried out along the same lines as done for I in the last section, so the details are left to the reader.

I is generated by I' and the inversion

$$x = (x_1, x_2, x_3) \mapsto -x = (-x_1, -x_2, -x_3). \tag{1}$$

But if $x^{I'}$ is a t-orbit ($t \in \mathbb{N}$) of I', so is $-x^{I'} = (-x)^{I'}$. The union of both these sets has the same property and is an orbit of I; hence we do not get anything new for I' instead of I.

4. Conclusion

We have shown that only two of the finite three-dimensional groups have 9-orbits (definition in Section 1), namely, the icosahedral groups I and I'. Furthermore, no 10-orbits of I or I' exist and among the 9-orbits there is—up to the choice of the representation of I—a unique set M which is optimal with respect to all polynomials of degree less than 12 in three variables.

This orbit contains 60 points and is most useful for integration on the sphere, for its allows to integrate all spherical harmonics of degree at most 9 exactly and minimizes the integration error for harmonics of 10 and 11 order.

APPENDIX

To illustrate the degree of approximation of integrals over the sphere by our formula we shall consider a numerical example the exact solution of which is known.

Let E be an ellipsoid with principal semi-axes a, b, c.

The surface of E consists of the points with Cartesian coordinates (ax, by, cz) where (x, y, z) represents an associated unit vector, i.e., a point of the unit sphere Ω_3.

The surface area F_E of E is given by

$$F_E = \int_{v \in \partial E} dO_v = \int_{(x,y,z) \in \Omega_3} \sqrt{b^2c^2x^2 + c^2a^2y^2 + a^2b^2z^2}\, dO_{(x,y,z)}. \tag{1}$$

The integral in (1) may be reduced further to the form

$$F_E = \pi a^2b^2c^2 \int_{t=-1}^{1} [a^2 - (a^2 - c^2)\,t^2]^{-3/2}[b^2 - (b^2 - c^2)\,t^2]^{-3/2} \times [a^2 + b^2 - (a^2 + b^2 - 2c^2)\,t^2]\, dt \tag{2}$$

(if $a \geqslant b \geqslant c$), or

$$F_E = 2\pi c^2 + 2\pi \int_{t=0}^{1} \frac{a^2b^2 - (a^2 - c^2)(b^2 - c^2)\,t^2}{\sqrt{(a^2 - (a^2 - c^2)\,t^2)(b^2 - (b^2 - c^2)\,t^2)}}\, dt, \tag{3}$$

which is equivalent to

$$F_E = 2\pi c^2 + \frac{2\pi bc^2}{\sqrt{a^2 - c^2}} F(\varphi, k) + 2\pi b \sqrt{a^2 - c^2}\, E(\varphi, k), \tag{4}$$

where

$$\cos\varphi = \frac{c}{a}; \qquad \sin\varphi = \frac{\sqrt{a^2 - c^2}}{a}, \tag{5}$$

$$k = \frac{a}{b}\sqrt{\frac{b^2 - c^2}{a^2 - c^2}}. \tag{6}$$

Here, E and F are incomplete elliptic integrals of the first and second kind, respectively (in Legendre's notation, cf., e.g., Abramowitz and Stegun [9]. We have calculated the surface areas of a number of ellipsoids with different a, b, c. The results are listed in Table II, where the "exact" values are found by Eqs. (4), (5), and (6). Furthermore our numerical integration method (averaging over the set M, Table I) is applied to (1). For comparison the results of averaging over the six vertices of a regular octahedron:

$$M' = \{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\} \tag{7}$$

TABLE II

Surface Area of An Ellipsoid with Principal Semi-axes a, b, c

a	b	c	Exact value (4)	Numerical value (M)	Relative error (10^{-6})	Numerical value (M')	Relative error (10^{-6})
1	1	1	12.56637	12.56637	0	12.56637	0
1	1	1.5	16.91822	16.92223	237	16.75516	−9873
1	1.1	1.2	15.18864	15.18907	28	15.16342	−1661
1	1.1	1.3	16.09793	16.09956	101	16.04307	−3408
1	1.1	1.4	17.01866	17.02229	213	16.92271	−5638
1	1.1	1.5	17.94935	17.95580	359	17.80236	−8189
1	1.2	1.2	16.12200	16.12156	−27	16.08495	−2298
1	1.2	1.3	17.06968	17.07033	38	17.00649	−3702
1	1.2	1.4	18.02976	18.03258	156	17.92802	−5643
1	1.2	1.5	19.00064	19.00673	320	18.84956	−7952
1	1.5	2	27.88644	27.91159	902	27.22714	−23643
1	2	3	48.88215	48.94900	1368	46.07669	−58680
1	3	5	108.62688	108.38537	−2223	96.34217	−113091
1	3	6	129.12957	129.02373	−820	113.09734	−124156
2	3	5	134.77518	135.12428	2590	129.85250	−36525
2	4	5	166.85563	166.81746	−229	159.17403	−46037
3	4	5	199.45506	199.54696	461	196.87314	−12945
3	4	6	231.24822	231.57445	1411	226.19467	−21853
3	5	8	345.82071	346.65038	2399	330.91443	−43104
3	6	8	396.81333	396.93240	300	376.99112	−49953
4	5	7	352.83158	353.12337	827	347.66959	−14630
4	5	8	393.86966	394.47189	1529	385.36870	−21583

Note. The exact values of the surface areas are found via Eq. (4), the numerical approximations by averaging over the optimal 9-design M (Table I) and the set M' (Eq. (7)) used in Abramowitz and Stegun [9, Eq. 25.4.65], respectively.

(recommended in Abramowitz and Stegun [9, formula 25.4.65]) are included as well. The relative errors of both approximations are given also. It is seen that the results of our method are much closer to the real areas (by a factor of ca. 10 to 100), the largest deviation amounting to less than 0.26 percent in a case where the integrand varies over quite a large range. The other formula give errors up to more than 12.4 percent in the cases which we investigated.

Similar conclusions hold for a lot of other examples which we do not discuss further.

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